Semisimple Hopf actions and factorization through group actions

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joint work with

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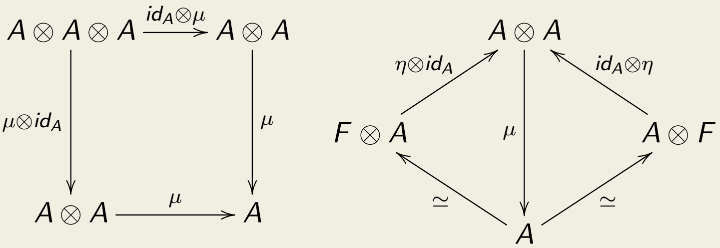
2017-06-15

Algebras

Let F be a field.

Definition

An F-algebra (A, μ, η) is an F-vector space A with linear maps μ and η such that



Coalgebras

Definition

A coalgebra (C, Δ, ϵ) is a F-vector space with maps Δ and ϵ, s.t.

$$\begin{CD} C @>\Delta>> C \otimes C \\ @V{\Delta}VV @VV{\Delta \otimes id_C}V \\ C \otimes C @>>{id_C \otimes \Delta}> C \otimes C \otimes C \end{CD}$$

$$\begin{array}{ccccc} F \otimes C & \xleftarrow{\epsilon \otimes id_C} & C \otimes C & \xrightarrow{id_C \otimes \epsilon} & C \otimes F \\ & \nwarrow & \uparrow{\scriptstyle \Delta} & \nearrow & \\ & & C & & \end{array}$$

Group algebras as Hopf algebras

Definition

For a coalgebra (C, Δ, ϵ) and an algebra (A, μ, η), $\mathrm{Hom}(C, A)$ is an F-algebra with convolution product:

$$(f * g)(c) = m \circ (f \otimes g)\Delta(c), \qquad \forall c \in C, f, g \in \mathrm{Hom}(C, A)$$

Definition

A Hopf algebra H is an F-algebra and F-coalgebra (H, Δ, ϵ), with Δ and ϵ unital algebra maps, and such that id_H has a convolution inverse S in $\mathrm{Hom}(H, H)$.

$H = F[G]$ with $\Delta(g) = g \otimes g$, $\epsilon(g) = 1$, and $S : H \to H$ with $S(g) = g^{-1}$, is a Hopf algebra.

H_8 (Kac-Paljutkin Hopf algebra) $\mathrm{char}(F) \neq 2$

H_8 is the algebra over F generated by x, y, and z subject to the following relations

$$\begin{aligned} x^2 = 1, \quad y^2 = 1, \quad xy = yx \\ z^2 = \frac{1}{2}(1 + x + y - xy), \quad zx = yz, \quad zy = xz. \end{aligned}$$

H_8 has a coalgebra structure with

$$\begin{aligned} \Delta(x) &= x \otimes x, & \epsilon(x) &= 1 \\ \Delta(y) &= y \otimes y, & \epsilon(y) &= 1 \\ \Delta(z) &= \tfrac{1}{2}(1 \otimes 1 + x \otimes 1 + 1 \otimes y - x \otimes y)(z \otimes z), & \epsilon(z) &= 1. \end{aligned}$$

H_8 becomes a Hopf algebra by setting $S(x) = x$, $S(y) = y$, and $S(z) = z$.

Hopf actions

Definition

Let H be a Hopf algebra and A be an algebra. A is an H-module algebra (H acts on A) if A is an H-module and

1) $h \cdot (ab) = \sum (h_1 \cdot a)(h_2 \cdot b)$;
2) $h \cdot 1_A = \epsilon(h) 1_A$,

for all $h \in H$, and $a, b \in A$.

Factors through a group action

Definition

Let H be a Hopf algebra acting on an algebra A. If there exists $I \subseteq \mathrm{Ann}_H(A)$ such that $H/I \cong F[G]$, the action of H on A *factors through a group action*.

H acts *inner faithfully* on A if there is no Hopf ideal $0 \neq I \subset \mathrm{Ann}_H(A)$.

Question

Question

Given an algebra A, are there Hopf actions which are not given by group actions?

Etingof-Walton's Theorem

Theorem (Etingof-Walton 2013, $\bar{F} = F$)

Any action of a semisimple, cosemisimple Hopf algebra on a commutative domain factors through a group action.

Cuadra-Etingof-Walton's Theorem

Corollary (Cuadra-Etingof-Walton 2014, $\bar{F} = F$)

Any action of a semisimple, cosemisimple Hopf algebra H on a division algebra D which is finite over its center Z such that

$$\gcd([D:Z], dimH!) = 1,$$

factors through a group action.

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Theorem (Cuadra-Etingof-Walton 2014, $\mathrm{char}(F) = 0$, $\bar{F} = F$)

Any action of a semisimple Hopf algebra H on the nth Weyl algebra factors through a group action.

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Ring of structure constants

Fix a basis $\{b_1, \ldots, b_n\}$ of H, then there exist constants

$$b_i b_j = \sum \mu_k^{ij} b_k, \qquad \Delta(b_k) = \sum \gamma_{ij}^k b_i \otimes b_j, \qquad S(b_i) = \sum \nu_j^i b_j.$$

$$t = \sum \tau_i b_i, \qquad t^* = \sum \tau_i^* b_i^*.$$

Suppose $A \simeq k\langle x_1, \ldots, x_m\rangle / \langle p_1, \ldots, p_m\rangle$ and H action given by

$$b_i \cdot \overline{x_j} = \overline{f_{ij}}, \qquad f_{ij} \in k\langle x_1, \ldots, x_m\rangle.$$

Define the *subring of structure constants* of F:

$$R = \left\langle \mu_k^{ij}, \gamma_{ij}^k, \nu_j^i, \tau_i, \tau_i^*, \epsilon(b_i), \text{coef}.p_i, \text{coef}.f_{ij} \right\rangle \subseteq F.$$

Hilbert-Rings

R is domain and a finitely generated $\mathbb{Z}$-algebra.

R is a *Hilbert ring* (O. Goldman 1951).

1. every prime ideal is the intersection of maximal ideals;
2. $R/\mathfrak{m}$ is a finite field, for every $\mathfrak{m} \in \mathrm{MaxSpec}(R)$.
3. "there exist enough maximal ideals":
   for every $q > 0$ there exists $X \subseteq \mathrm{MaxSpec}(R)$ with

$$char(R/\mathfrak{m}) > q, \qquad \forall \mathfrak{m} \in X, \qquad \text{and} \bigcap_{\mathfrak{m}\in X} \mathfrak{m} = 0.$$

Reduction mod p

$$H \quad \rightsquigarrow \quad H_R := \bigoplus Rb_i \quad \rightsquigarrow \quad H_{\mathfrak{m}} := H_R \otimes_R R/\mathfrak{m}$$

leading to a semisimple, cosemisimple Hopf algebra $H_{\mathfrak{m}}$ over the finite field $R/\mathfrak{m}$, for all $\mathfrak{m} \in \mathrm{MaxSpec}(R)$

$$A \rightsquigarrow A_R := R\langle x_1, \ldots, x_m\rangle / \langle p_1, \ldots, p_m\rangle \rightsquigarrow A_{\mathfrak{m}} := A_R \otimes_R R/\mathfrak{m}$$

with $H_{\mathfrak{m}}$ acting on $A_{\mathfrak{m}}$.

Hopf actions factor through group actions

Theorem (Lomp-P, 2015, $\mathrm{char}(F) = 0$, $\bar{F} = F$)

Let H be a semisimple Hopf algebra acting on finitely presented algebra A, such that there exists $q > 0$ and for all $\mathfrak{m} \in \mathrm{MaxSpec}(R)$ with $char(R/\mathfrak{m}) > q$:

1. *$A_{\mathfrak{m}}$ is a Noetherian domain with divison ring of fractions $D_{\mathfrak{m}}$.*
2. *$D_{\mathfrak{m}}$ is finite over its center $C_{\mathfrak{m}}$ and*

$$\gcd\left([D_{\mathfrak{m}} : C_{\mathfrak{m}}], dim(H)!\right) = 1.$$

Then the action of H factors through a group action.

Applications

Corollary

The Theorem applies to the following classes of algebras A, because in each case $[D_{\mathfrak{m}} : C_{\mathfrak{m}}]$ is a power of p if $char(R/\mathfrak{m}) = p$ (then choose $q = dim(H)$):

1. $A = A_n(F)$;
2. $A = U(\mathfrak{g})$;
3. $A = F[x_0][x_1, \delta_1][x_2, \delta_2] \cdots [x_m, \delta_m]$;

Question

Question

Given an algebra A, are there semisimple Hopf algebra actions on A which are not group actions?

Constructing a semisimple Hopf algebra

Example

Let $n > 1$, $q \in F$ a primitive n-th root of unity and $R = F[G]$ for

$$G = \langle x, y | y^n = x^n = 1 \text{ and } xy = yx \rangle.$$

Constructing a semisimple Hopf algebra

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Let $n > 1$, $q \in F$ a primitive n-th root of unity and $R = F[G]$ for

$$G = \langle x, y | y^n = x^n = 1 \text{ and } xy = yx \rangle.$$

Let $\sigma \in \mathrm{Aut}(R)$ with $\sigma(x) = y$ and $\sigma(y) = x$. Then $A = R[z; \sigma]$ extends the bialgebra structure of R with $\epsilon(z) = 1$ and

$$\Delta(z) = J(z \otimes z) \qquad \text{and} \qquad J = \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} q^{-ij} x^i \otimes y^j.$$

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Moreover $H_{2n^2} = R[z; \sigma] / \langle z^2 - t \rangle$ is a semisimple Hopf algebra of dimension $2n^2$ where $t = \frac{1}{n} \sum_{j=0}^{n-1} \sum_{i=0}^{n-1} q^{-ij} x^i y^j$ and $S(z) = z$.

H_8 as a quotient of an Ore extension

Take $n = 2$ and $q = -1$. Also $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \langle x, y \mid x^2 = 1 = y^2, xy = yx \rangle$, the element J is given by

$$J = \frac{1}{2}(1 \otimes 1 + x \otimes 1 + 1 \otimes y - x \otimes y).$$

Then

$$H_8 = R[z;\sigma]/\langle z^2 - t \rangle,$$

where $\frac{1}{2}(1 + x + y - xy)$.

Constructing an inner faithful action

Let $M = (m_{ij}) \in M_{r \times r}(F^{\times})$ be a square matrix of size r such that $m_{ii} = m_{ij}m_{ji} = 1$. Let $A_M = F_M[u_1, \ldots, u_r]$ be the *quantum polynomial algebra*, i.e., the associative F-algebra generated by $u_1, \ldots, u_r$ subject to the relations

$$u_i u_j = m_{ij} u_j u_i, \quad 1 \leq i, j \leq r.$$

Constructing an inner faithful action

Theorem (Lomp-P, 2017)

For any $n, r > 1$, primitive nth root of unity q, integers $0 \leq a_i, b_i \leq n-1$, for $i \in \{1, \ldots, r\}$, permutation $\tau \in S_r$, and matrix $M = (m_{ij}) \in M_{r \times r}(\mathbb{C})$ such that $m_{ij} = m_{ij} m_{ji} = 1$ and

$$m_{\tau(i)\tau(j)} = q^{a_{\tau(j)} b_{\tau(i)} - a_{\tau(i)} b_{\tau(j)}} m_{ij},$$

for all i, j, there exists an action of H_{2n^2} on the quantum polynomial algebra A_M with

$$x \cdot u_i = q^{a_i} u_i, \quad y \cdot u_i = q^{b_i} u_i, \quad z \cdot u_i = u_{\tau(i)}.$$

Theorem (Lomp-P, 2017)

If for all $i, j \in \{0, \dots, n-1\}$, with $(i, j) \neq (0, 0)$, there exists $k \in \{1, \dots, r\}$ such that

$$ia_k \not\equiv -jb_k (\mathrm{mod} n), \quad (*)$$

then the action of H_{2n^2} on A_M is inner faithful.

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Condition $(*)$ is satisfied if some 2×2 minor of

$$\begin{pmatrix} a_1 & a_2 & \cdots & a_r \\ b_1 & b_2 & \ldots & b_r \end{pmatrix}$$

has an invertible determinant in $\mathbb{Z}_n$.

Actions on the Quantum Plane

Let $0 \neq p \in F$ and consider the matrix

$$M = \begin{pmatrix} 1 & p^{-1} \\ p & 1 \end{pmatrix} \in M_{2\times 2}(F)$$

The *quantum plane* is the quantum polynomial algebra $A_M = F_M[u, v]$ with two generators.

An action of H_{2n^2} on $F_p[u, v]$

Consider the Hopf algebras H_{2n^2}. For each n, these Hopf algebras act on $A = \mathbb{C}_p[u, v]$ with $p^2 = q$. The action is given by:

$$\begin{aligned} x \cdot u &= qu, & y \cdot u &= u, & z \cdot u &= v, \\ x \cdot v &= v, & y \cdot v &= qv, & z \cdot v &= u, \end{aligned}$$

which corresponds to $\tau = (12) \in S_2$ and the matrix

$$B = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_{2\times 2}(\mathbb{Z}_2)$$

Since the matrix B is invertible in $\mathbb{Z}_2$, this action is inner faithful.

H_8 acting on $\mathbb{C}_p[u, v]$

Theorem (Lomp-P, 2017)

Let $1 \neq p \in \mathbb{C}^\times$. If there is a Hopf action of H_8 on the quantum plane $A = \mathbb{C}_p[u, v]$ such that $z \cdot u = v$ and $z \cdot v = u$, then this action is inner faithful and $p^2 = -1$.

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Example

Let $A = \mathbb{C}_{-1}[u, v]$ be the quantum plane. Then H_8 acts on A as follow:

$$x \cdot u = u, \qquad y \cdot u = -u, \qquad z \cdot u = u,$$
$$x \cdot v = v, \qquad y \cdot v = -v, \qquad z \cdot v = -v.$$

This action is inner faithful.

Thank you

Thank you for your attention.